ENTROPY OF SOME INNER AUTOMORPHISMS OF THE HYPERFINITE II_1-FACTOR.

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Abstract.

It is shown that the entropy of an inner automorphism $\mathrm{Ad}\, u$ of the hyperfinite II_1-factor is zero if the unitary operator u belongs to a Cartan subalgebra.

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While the entropy in the sense of [4] of automorphisms of the hyperfinite II_1-factor R has been computed for several outer automorphisms, there are only few partial results on inner automorphisms, see [1]. If T is a nonsingular measure preserving ergodic transformation of a probability space (X, B, μ) then T defines a unitary operator U_T on $L^2(X,\mu)$ by $(U_T f)(x) = f(T^{-1}x)$, $f \in L^2(X,\mu)$, $x \in X$. Furthermore the von Neumann algebra generated by $L^\infty(X,\mu)$ and U_T equals R. It is immediate from the definition of entropy that the entropy of the inner automorphism $\mathrm{Ad}\, U_T$ of R satisfies

$$h(\mathrm{Ad}\, U_T) \geq h(\mathrm{Ad}\, U_T | L^\infty(X,\mu)) = h(T),$$

hence in particular $h(\mathrm{Ad}\, U_T) > 0$ whenever $h(T) > 0$. In the present note it will be shown that in this case U_T cannot belong to a Cartan subalgebra (also called regular masa) of R, i.e. the normalizer of the maximal abelian subalgebra generates R. This result is immediate from

Theorem. Let u be a unitary operator contained in a Cartan subalgebra of the hyperfinite II_1-factor R, and $\mathrm{Ad} u$ the inner automorphism of R defined by u. Then the entropy $h(\mathrm{Ad}\, u) = 0$.

Thus the entropy of $\mathrm{Ad} u$ for u unitary in R depends essentially on the position in R of the abelian von Neumann subalgebra generated by u.

I am indebted to O. Bratteli and E. Bedos for helpful comments on the following lemma and the corollary respectively.

Lemma. Let u be a unitary operator on a Hilbert space H. Let $k \in \mathbf{N}$ and $z_r = \exp(i2\pi k^{-2} r)$ for $r \in \{0,1,2,\ldots,k^2\}$. Let p_r denote the spectral projection $p_r = \chi_{[z_{r-1}, z_r)}(u)$ of u, and let v denote the unitary operator $v = \sum_{r=1}^{k^2} z_r p_r$. Then

$$\|u^j - v^j\| < 2\pi/k \quad \text{for} \quad j \in \{1,2,\ldots,k\}.$$

Proof. Straightforward estimates using spectral theory show

$$\|u - v\| < 2\pi/k^2.$$

Since $u^j - v^j = (u-v)\sum_{i=0}^{j-1} u^{j-i-1} v^i$, we have

$$\|u^j - v^j\| \leq \|u - v\| j \leq j 2\pi/k^2 \leq 2\pi/k \quad \text{for} \quad j \in \{1,\ldots,k\}. \qquad \text{QED.}$$

Proof of Theorem. By the Connes-Feldman-Weiss theorem [3], [5], all Cartan subalgebras of R are conjugated by an automorphism of R. Since $\mathrm{Ad}\alpha(u) = \alpha \circ \mathrm{Ad}\, u \circ \alpha^{-1}$ for $\alpha \in \mathrm{Aut}\, R$ we have $h(\mathrm{Ad}\, \alpha(u)) = h(\mathrm{Ad}\, u)$, hence we may assume u belongs to a Cartan subalgebra of the form $D = \bigotimes_{i=1}^{\infty} D_{n_i}$, where D_{n_i} is the diagonal algebra in $M_{n_i}(\mathbf{C})$, and $R = \bigotimes_{i=1}^{\infty} M_{n_i}(\mathbf{C})$, taken with respect to the tensor product of the normalized traces $\frac{1}{n_i} Tr_{n_i}$. Let

$$P_m = (\bigotimes_{i=1}^{m} M_{n_i}(\mathbf{C})) \otimes \mathbf{C} \subset \mathbf{R}, \quad m \in \mathbf{N}.$$

Then $(P_m)_{m\in\mathbf{N}}$ is an increasing sequence of finite type I subfactors of R with union weakly dense in R. Thus by the Kolmogoroff-Sinai theorem [4, Thm. 2] we have

$$h(\mathrm{Ad}\, u) = \lim_{m\to\infty} H(P_m, \mathrm{Ad}\, u),$$

where

$$H(P_m, \mathrm{Ad}\, u) = \lim_{k\to\infty} \frac{1}{k} H(P_m, \mathrm{Ad}\, u(P_m), \ldots, \mathrm{Ad}\, u^{k-1}(P_m)),$$

see [3]. Fix $m \in \mathbf{N}$, and put $P = P_m$. Then there is $n \in \mathbf{N}$ so that $P \cong M_n(\mathbf{C})$. Then with D_n the diagonal of P we have

$$D = D_n \otimes A, \quad \text{with} \quad A = \bigotimes_{i=m+1}^{\infty} D_{n_i}.$$

Since $u \in D$

$$u = \sum_{i=1}^{n} e_i \otimes u_i$$

with $e_1, \ldots, e_n$ the minimal projections in D_n, and u_i, $1 \le i \le n$, a unitary operator in A.

Let $\epsilon > 0$. By [4, Thm. 1] there is $\delta > 0$ such that if $Q, M \subset R$ are finite dimensional $*$-subalgebras with $Q \cong P$ and $Q \overset{\delta}{\subset} M$, then the relative entropy $H(Q|M) < \epsilon/2$. Choose $k \in \mathbf{N}$ so large that $k > 4\pi/\delta$ and

$$\frac{1}{k}(\log n + 2n \log k) < \epsilon/2.$$

For each $i \in \{1, \ldots, n\}$ choose a unitary operator $v_i \in A$ of the form $v_i = \sum_{r=1}^{k^2} z_{ir} p_{ir}$ as in lemma, so that $\|v_i^j - u_i^j\| < 2\pi/k$ for $j \in \{1, \ldots, k\}$. Let B denote the von Neumann subalgebra of A generated by the projections p_{ir}, $i \in \{1,\ldots,n\}$, $r \in \{1,\ldots,k^2\}$. Then B has at most $(k^2)^n$ minimal projections, hence its entropy $H(B)$ satisfies

$$H(B) \leq \log k^{2n} = 2n \log k.$$

Furthermore, if $v = \sum e_i \otimes v_i$ then $v \in P \otimes B$, and

$$\|u^j - v^j\| = \|\sum_{i=1}^{n} e_i \otimes (u_i^j - v_i^j)\| = \max \|u_i^j - v_i^j\| < 2\pi/k,$$

for $j \in \{1,\ldots,k\}$. Thus, since $\|x\|_2 \leq \|x\|$ for $x \in R$,

$$\operatorname{Ad} u^j(P) \overset{4\pi/k}{\subset} P \otimes B,$$

hence by choice of k

$$H(\operatorname{Ad} u^j(P)|P \otimes B) < \epsilon/2 \quad \text{for} \quad j \in \{0,1,\ldots,k\}.$$

It follows from properties (C) and (F) of the entropy function H, see [4], that

$$\begin{aligned}
\frac{1}{k}H(P, \operatorname{Ad} u(P),\ldots,\operatorname{Ad} u^{k-1}(P)) &\leq \frac{1}{k}(H(P \otimes B) + \sum_{j=0}^{k-1} H(\operatorname{Ad} u^j(P)|P \otimes B)) \\
&< \frac{1}{k}(H(P) + H(B) + k\epsilon/2) \\
&\leq \frac{1}{k}(\log n + 2n \log k) + \epsilon/2 \\
&< \epsilon.
\end{aligned}$$

Since ϵ is arbitrary and this holds for all sufficiently large k, $H(P, \operatorname{Ad} u) = 0$, and hence $h(\operatorname{Ad} u) = 0$. QED

Let u be a unitary operator and A an injective von Neumann algebra both acting on the same Hilbert space H. Suppose $\operatorname{Ad} u$ restricts to an ergodic, properly outer automorphism of A and that there is a faithful normal invariant finite trace on A. Then by [6.22.2] the von Neumann algebra generated by A and u is the crossed product $A \times_{\operatorname{Ad} u} \mathbf{Z}$, hence is by [2,Prop.6.8] and [6,22.6] the hyperfinite II_1-factor R. Let B denote the von Neumann algebra generated by u. Then B is a masa in R. Indeed, suppose $x \in R \cap B'$. Then x has a Fourier expansion $x = \sum_{n=-\infty}^{\infty} a_n u^n$ with $a_n \in A$, and the series converges in L^2-norm. Since $x = uxu^{-1}$ we get $\sum a_n u^n = \sum u a_n u^{-1} u^n$, hence by uniqueness of Fourier coefficients, $a_n = u a_n u^{-1}$, so by ergodicity, $a_n \in \mathbf{C}$, hence $x \in B$, and B is a masa as asserted. Suppose furthermore that there is a family V of unitaries generating A such that $vBv^* = B$ for all $v \in V$. Since V and B generate R the normalizer of B generates R, so that B is a Cartan subalgebra. We thus have the following corollary of the Theorem.

Corollary. Let u be a unitary operator and A an injective von Neumann algebra both acting on the Hilbert space H. Suppose

(1) Ad u is an ergodic properly outer automorphism of A with a faithful normal invariant finite trace.

(2) A is generated by a family V of unitaries in the normalizer of the von Neumann algebra generated by u.

Then the Neumann algebra generated by A and u is the hyperfinite II_1-factor R, and the entropy $h(\mathrm{Ad}\, u) = 0$ of Ad u as an automorphism of R. In particular the entropy $h(\mathrm{Ad}\, u|A) = 0$

In the special case when if T_θ is the irrational rotation on the circle by an angle θ, and $u = U_{T_\theta}$ the corresponding unitary on $L^2(\mathrm{T})$ we obtain the above situation with v the multiplication operator $(vf)(\phi) = e^{i\phi}f(\phi)$ for $f \in L^2(\mathrm{T})$, since then $uv = e^{i\theta}vu$. In this case $A = L^\infty(\mathrm{T})$ is the von Neumann algebra generated by v, and the family V consists of v alone. We thus have $h(\mathrm{Ad}\, U_{T_\theta}) = 0$, hence in particular the classical result that $h(T_\theta) = 0$.

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